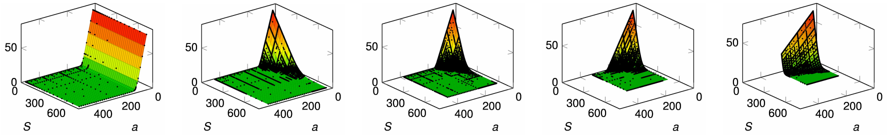
PRMIA Munich: Workshop on Variable Annuities

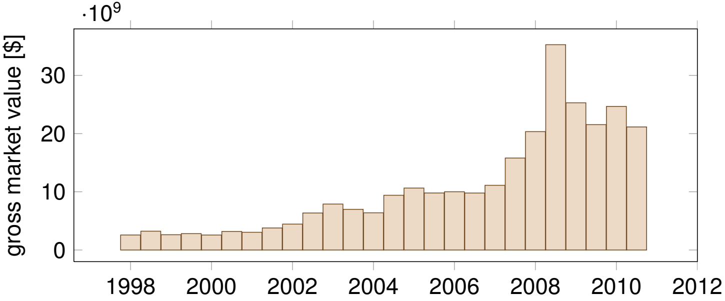
A High-Dimensional PDE-Based Approach for Capital Market Guarantees

Stefanie Schraufstetter

3. April 2012



Motivation: Development of the Financial Market



total volume of all OTC derivates in the G10 countries and Switzerland

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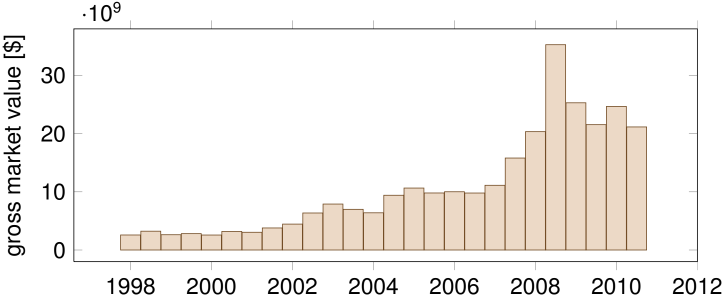

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problems:

- increasing variety
- increasing complexity

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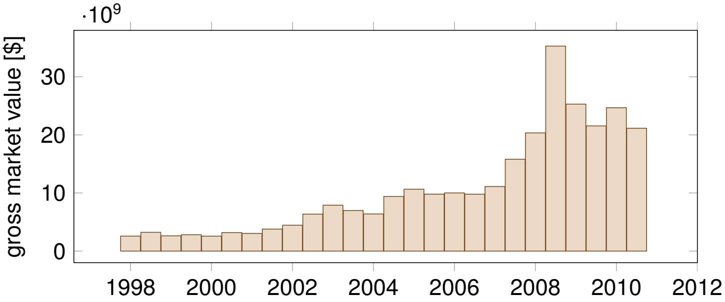

total volume of all OTC derivates in the G10 countries and Switzerland

problems:

- increasing variety
- increasing complexity

→ need for a unique software for evaluation of financial derivatives

Challenge „Pricing Framework“

Basic idea: model driven architecture

- unique product specification
- automated evaluation

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Modeling Language *ThetaML*:

- virtual timing model
- mainly 3 basic effects:
  - waiting
  - transacting
  - deciding
- independent of pricing algorithm

The Pricing Framework

```
model asianOption
  import S "underlying"
  import K "strike"
  export V "option price"

  a=0
  loop j=1:n
    Theta 1/n
    a = a+S
  end
  V=max(a/n-K,0)
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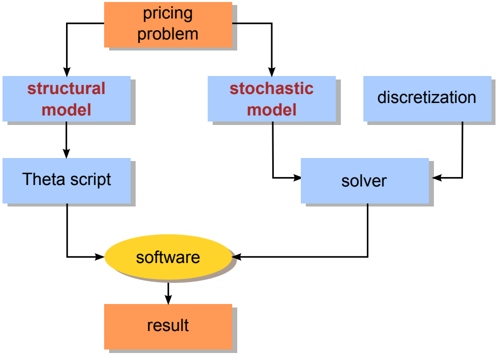

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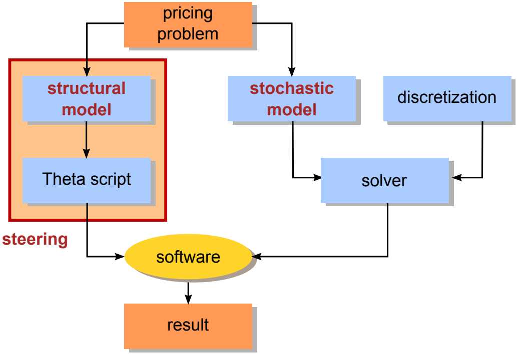

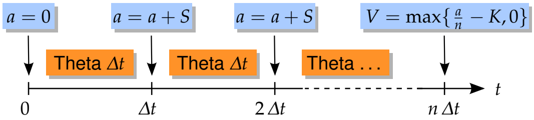

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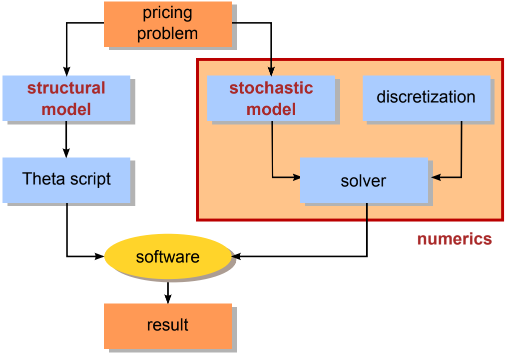

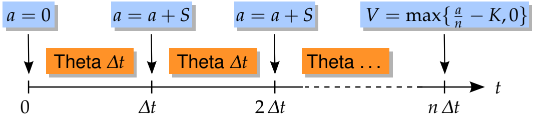

MC vs. PDE – State of the Art

MC:

- independent of dimension ("nearly")
- (theoretical) slow speed of convergence: $\mathcal{O}(N^{\frac{1}{2}})$
- different accelerating techniques, e.g.
  - quasi-MC method
  - multi-level MC
- estimation of conditional expectation difficult ($\rightarrow$ least-squares MC!)

PDE:

- (theoretical) high speed of convergence
- strongly dimension-dependent: $\mathcal{O}(N^d)$ dofs!
- typically complex algorithms
- well-suited for hedging
- topic of current research

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THETARIS
Engineering in Finance

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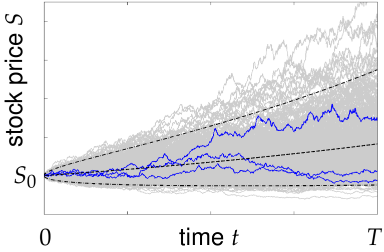
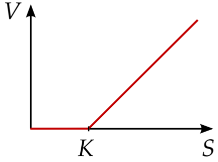
Option Pricing with PDEs

underlying: stochastic process, e.g. geometric Brownian motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad W_t \sim \mathcal{N}(0, t)$$





option price at
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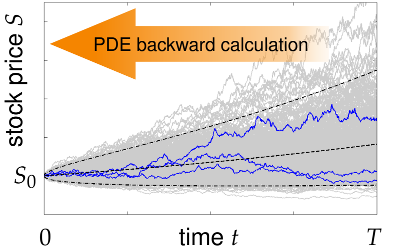

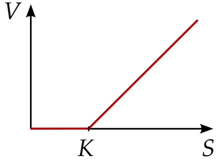

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From the Stock Price to the Option Value

1. Assume an underlying following a geometric Brownian motion:
   $dS_t = \mu S\, dt + \sigma S\, dW_t$
2. Apply Itô's lemma
3. Eliminate stochastic term by considering a risk-free portfolio
   $\Pi = V - \Delta \cdot S \qquad \Rightarrow d\Pi = r\Pi\, dt$

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Itô's lemma

Let S_t follow the SDE

$$dS_t = a(t, S_t)\, dt + b(t, S_t)\, dW_t$$

with a Wiener process W_t and $V = f(t, S_t)$ be twice differentiable. Then, the stochastic process for V is given by

$$dV = \left(\frac{\partial V}{\partial t} + a(t, S_t) \frac{\partial V}{\partial S} + (b(t, S_t))^2 \frac{\partial^2 V}{\partial S^2} \right) dt + b(t, S_t) \frac{\partial V}{\partial S}\, dW_t.$$

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Black-Scholes equation (1D)

$$\frac{\partial V(t,S)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(t,S)}{\partial S^2} + rS\frac{\partial V(t,S)}{\partial S} - rV(t,S) = 0$$

Notation:

V	option price	σ	volatility
S	stock price	r	risk-free rate

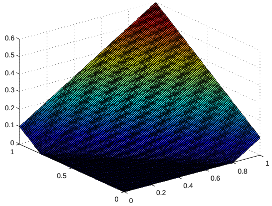
The Multi-Dimensional Black-Scholes Equation

Black-Scholes Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{d} (r - \delta_i) S_i \frac{\partial V}{\partial S_i} - rV = 0$$

final condition, e.g. for a call option:

$$V(\mathbf{S}, T) = \max\{\frac{1}{d}\sum_{i=1}^{d} S_i - K, 0\}$$



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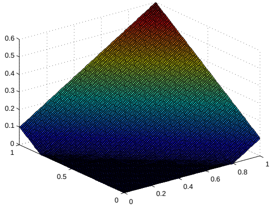


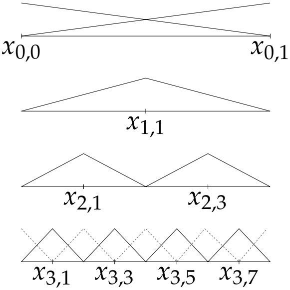
boundary condition:

$$V(\bar{\mathbf{S}}, t) = e^{-r(T-t)} V(\bar{\mathbf{S}}, T) \quad \forall \bar{\mathbf{S}} \in \partial\Omega$$

with $\Omega := [S_{1,\text{min}}, S_{1,\text{max}}] \times \ldots \times [S_{d,\text{min}}, S_{d,\text{max}}]$

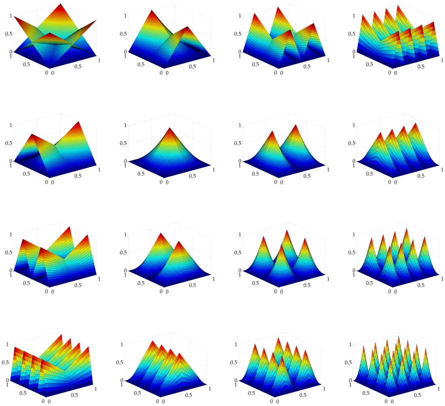
Discretization with Sparse Grids

hierarchical basis 1D



$\varphi_{l,i}(x)$: **function of level** l
with index i

hierarchical basis 2D



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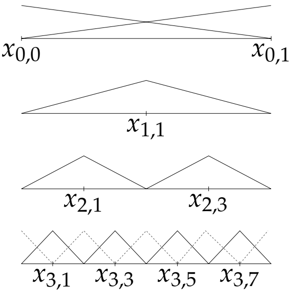
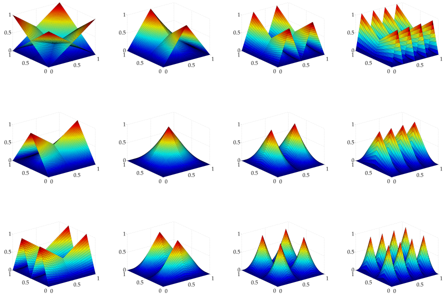


$\varphi_{l,i}(x)$: **function of level l with index i**

hierarchical basis 2D

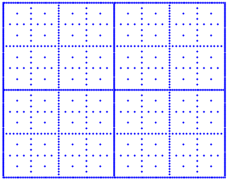


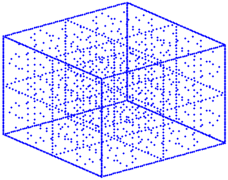
defined via tensor product:

$$\varphi_{\mathbf{l},\mathbf{i}}(\mathbf{x}) = \prod_k \varphi_{l_k,i_k}(x_k)$$

Discretization with Sparse Grids

sparse grid 2D/3D





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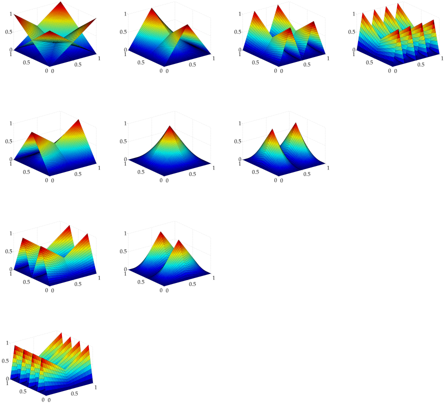
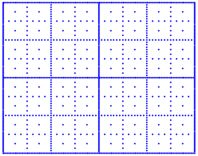


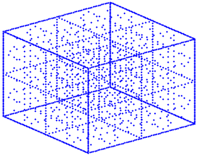
Discretization with Sparse Grids

Sparse grids vs. full grids:

	full grid	sparse grid
number of points	$\mathcal{O}(n^d)$	$\mathcal{O}(n(\log n)^{d-1})$
approximation accuracy	$\mathcal{O}(h^2)$	$\mathcal{O}(h^2(\log \frac{1}{h})^{d-1})$

with mesh width h and $n = 1/h$





Solving PDEs with Sparse Grids

‘standard’ approach:

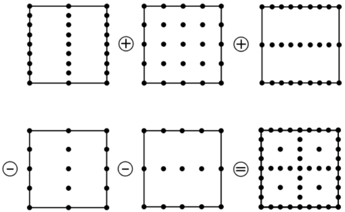
combination technique

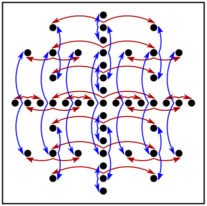
- extrapolation of solutions on smaller full grids
- simple to implement
- no spatial adaptivity

our approach:

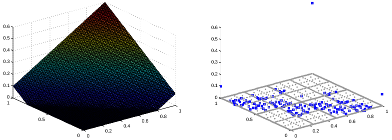
direct sparse grid approach

- works directly on the hierarchical basis
- complex algorithms
- spatial adaptivity

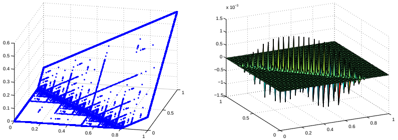




Why Spatial Adaptivity?



hierarchical coefficients



discretization error $\approx 10^{-4}$

- higher efficiency:
  only few surplusses
  contribute significantly

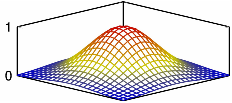
- higher accuracy:
  full-grid like refinement
  necessary to reduce
  discretization error
  at kinks, jumps, etc.

Results for a European Basket Option

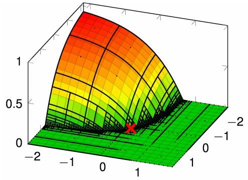
Comparison of degrees of freedom: non-adaptive vs. adaptive

dim.	accuracy (L^∞)	non-adaptive	adaptive		adaptive, weighted	
2	0.0002	98 305	13 402	(13.6%)	9 202	(9.4%)
3	0.0050	274 431	43 589	(15.9%)	28 488	(10.4%)
4	$\approx$ 0.0010	–	91 039	–	16 670	–
5	$\approx$ 0.0002	–	–	–	89 704	–

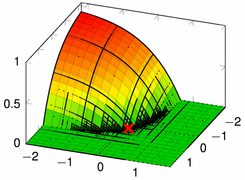
weighted adaptivity:



Gaussian function



non-weighted



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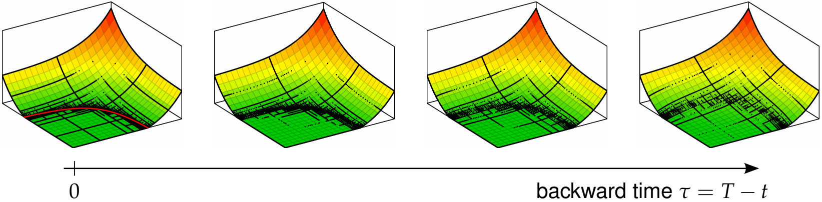
dynamic realization:



Automated Evaluation Algorithm

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   - lexical, syntactic, semantic analysis

   → deriving the operator sequence
2. forward estimation of domain size
3. backward calculation

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```
model     →  MODEL head EOC body END
body      →  operator | operator body
operator  →  IF formula THEN formula
             ELSE formula END |
             WAIT formula |
             LOOP formula EOC body END |
             assignop EOC | ...
formula   →  NUMBER | LITERAL |
             formula PLUS formula | ...
assignop  →  LITERAL ASSIGN formula
...
```

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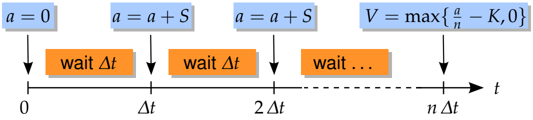

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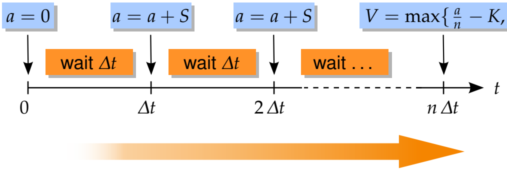

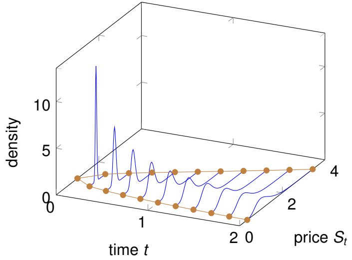

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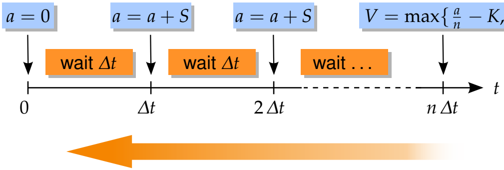

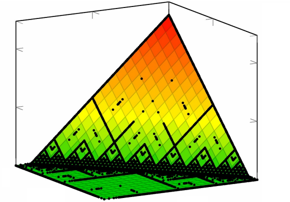

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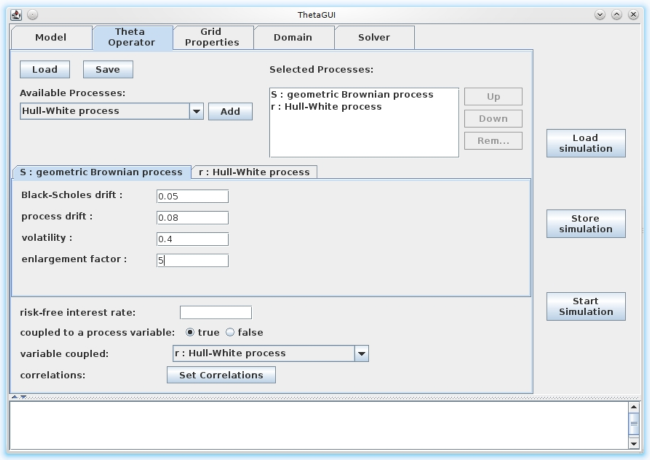
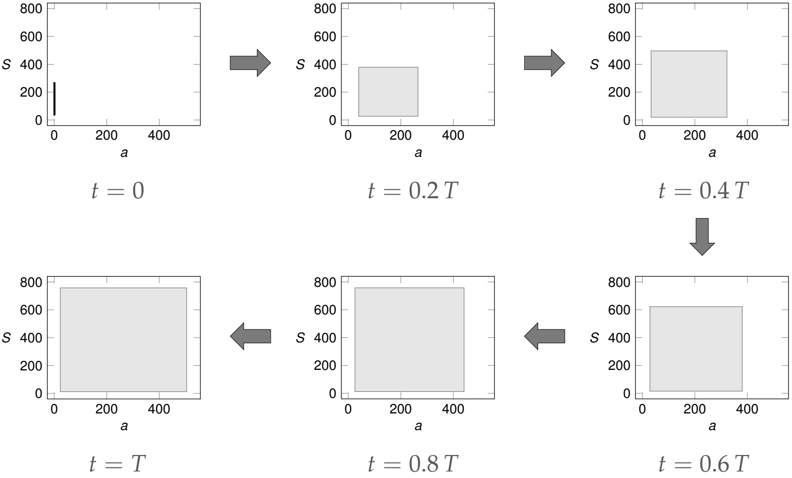
GUI: Model Parametrization



GUI $\Longleftrightarrow$ XML file with parametrization $\Longleftrightarrow$ framework

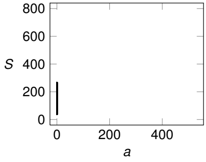
Calculation Cycle of an Asian Option

1. forward estimation of domain size

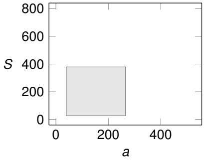


Calculation Cycle of an Asian Option

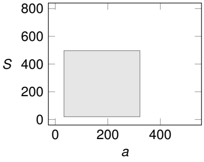
2. PDE backward calculation



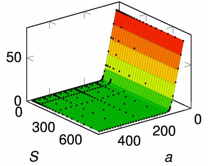
$t = 0$



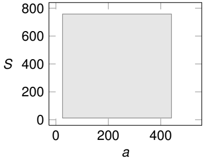
$t = 0.2\,T$



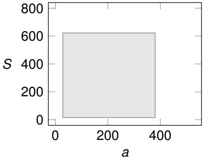
$t = 0.4\,T$



$t = T$



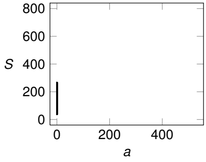
$t = 0.8\,T$



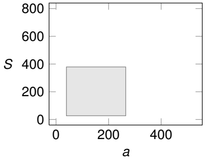
$t = 0.6\,T$

Calculation Cycle of an Asian Option

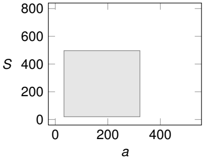
2. PDE backward calculation



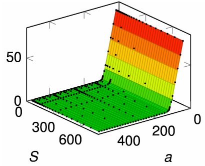
$t = 0$



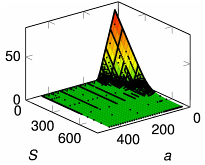
$t = 0.2\,T$



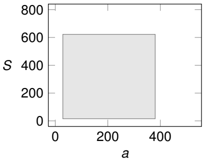
$t = 0.4\,T$



$t = T$



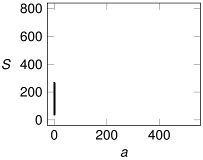
$t = 0.8\,T$



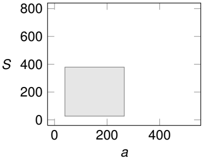
$t = 0.6\,T$

Calculation Cycle of an Asian Option

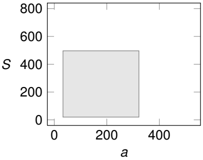
2. PDE backward calculation



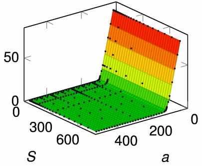
$t = 0$



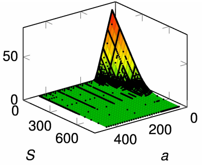
$t = 0.2\,T$



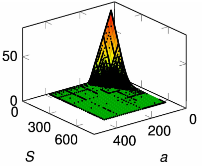
$t = 0.4\,T$



$t = T$



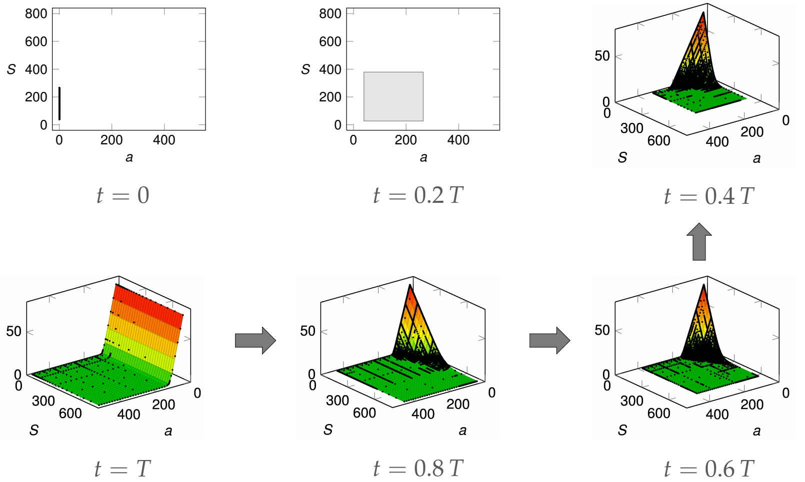
$t = 0.8\,T$



$t = 0.6\,T$

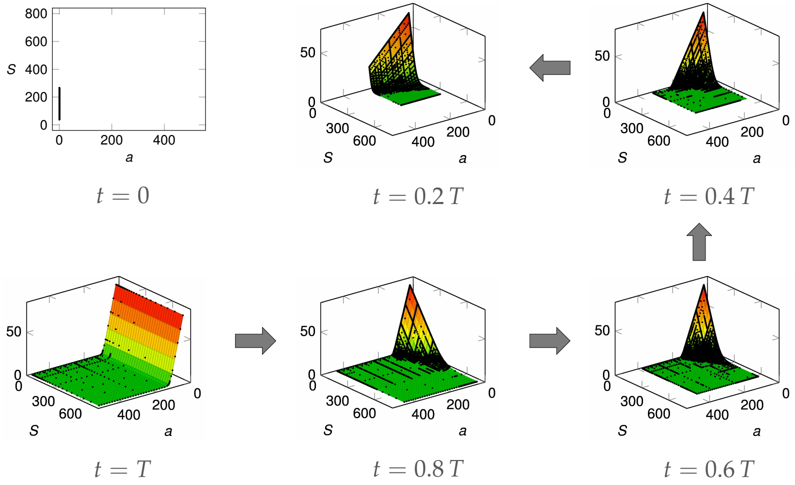
Calculation Cycle of an Asian Option

2. PDE backward calculation



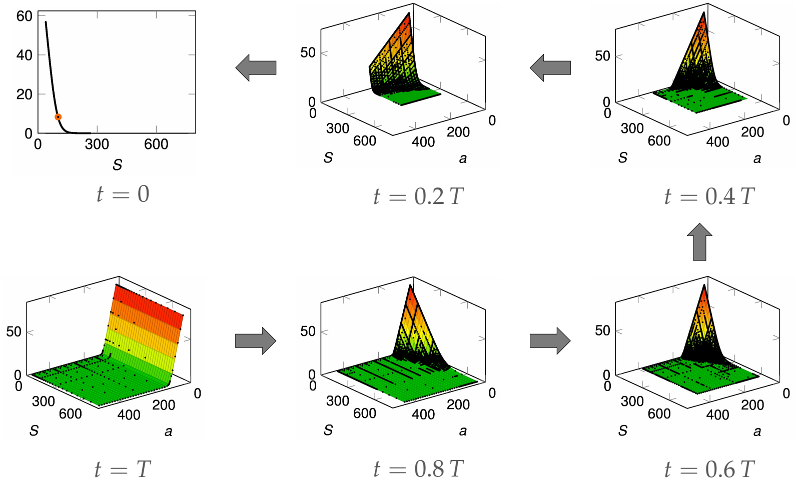
Calculation Cycle of an Asian Option

2. PDE backward calculation

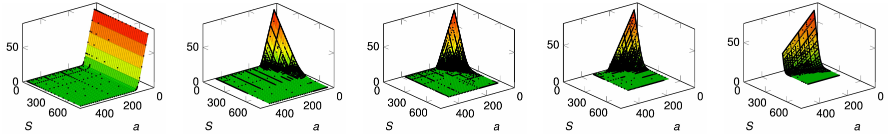


Calculation Cycle of an Asian Option

2. PDE backward calculation

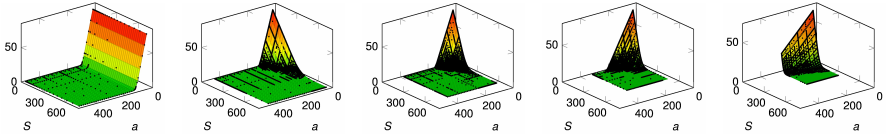


Asian Option – Results



max.	non-adaptive			adaptive		
level	∅ dofs	price	cpu time	∅ dofs	price	cpu time
7	1 281	8.13993	11	1 929	8.23363	19
8	2 727	8.11502	13	2 842	8.19200	28
9	5 357	8.17599	27	3 686	8.18145	37
10	10 703	8.17792	57	4 224	8.18027	43
11	21 543	8.17820	156	4 653	8.17958	50
12	43 758	8.17883	411	5 026	8.17951	55
13	89 284	8.17895	1 119	5 494	8.17956	64
14	181 117	8.18100	3 208	6 126	8.17943	69
15	359 173	8.18088	7 604	6 937	8.17948	84
MC reference value (10^6 paths): 8.18074						

Asian Option – Results



max. level	non-adaptive ∅ dofs	non-adaptive price	non-adaptive cpu time	adaptive ∅ dofs	adaptive price	adaptive cpu time
7	1 281	8.13993	11	1 929	8.23363	19
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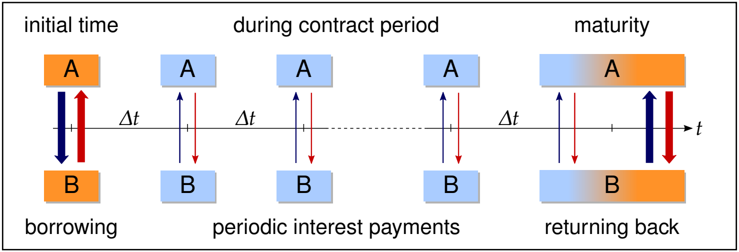
MC reference value (10^6 paths): 8.18074 $\pm 2 \cdot 10^{-3}$

Cross Currency Swap



variable	model	stochastic process
domestic interest rate r_d	Ho-Lee	$dr_d(t) = (\mu_{r_d} + \sigma_{r_d}^2 t)\, dt + \sigma_{r_d}\, dW_1(t)$
foreign interest rate r_f	Ho-Lee	$dr_f(t) = (\mu_{r_f} + \sigma_{r_f}^2 t)\, dt + \sigma_{r_f}\, dW_2(t)$
FX rate s	FX process	$ds(t) = (r_d(t) - r_f(t)) \cdot s(t)\, dt + \sigma_s \cdot s(t)\, dW_3(t)$

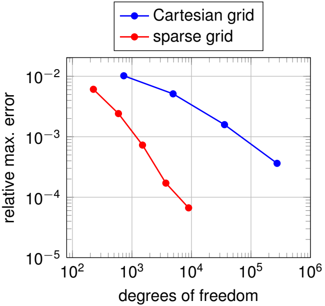
Cross Currency Swap

Steps for Evaluation:

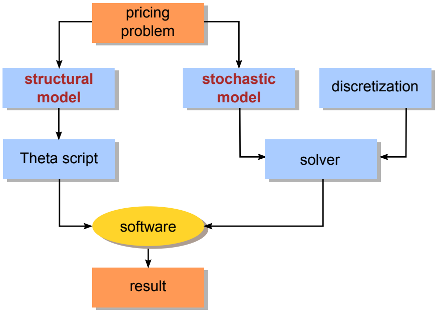
1. modeling the structure in form of a Theta script
2. specification of stochastic processes
3. specification of coupling of interest rate
4. start of automated evaluation

Numerical aspects:

- 3-dimensional problem
- smooth payoff function
- non-adaptive grid sufficient



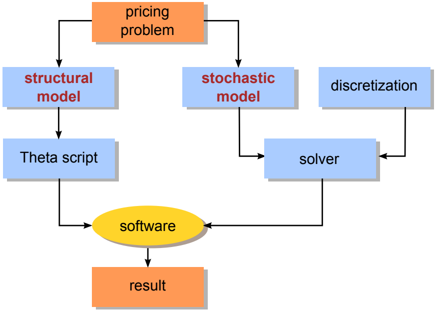
The Pricing Framework at a Glance



Financial products/strategies

- ✓ options with up to 6 underlyings
- ✓ swaps
- ✓ variable annuities
- ✓ hedging

The Pricing Framework at a Glance



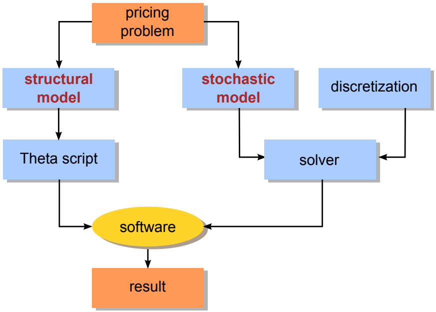
Financial products/strategies

- ✓ options with up to 6 underlyings
- ✓ swaps
- ✓ variable annuities
- ✓ hedging

numeric implementation

- ✓ higher-dimensional
- ✓ adaptive
- ✓ parallel & efficient

The Pricing Framework at a Glance



Financial products/strategies

- ✓ options with up to 6 underlyings
- ✓ swaps
- ✓ variable annuities
- ✓ hedging

numeric implementation

- ✓ higher-dimensional
- ✓ adaptive
- ✓ parallel & efficient

framework

- ✓ unique product specification
- ✓ automated evaluation
- ✓ modular structure

Thank you for your attention!

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